Differential equations of the first-order (theory)

Differential equation that separates the variables

$y` = f(x)\, g(y) \Leftrightarrow \frac{dy}{dx} = f(x)g(y) \Leftrightarrow \frac{dy}{g(y)} = f(x)dx \Leftrightarrow \int \frac{dy}{g(y)} = \int f(x)dx + c$ $\longrightarrow$ general integral

If there is **b** so that **g (b) = 0** then **y = b** is solution.

Homogeneous differential equations

$y`=f(\frac{y}{x})$ Can be solved by replacement $\frac{y}{x} = z$ $\longrightarrow$ $y`= z+xz`$. After replacement this differential equation is reduced to differential equation that separates the variables.

For $x=0$ $(y \neq 0)$ if there is z_k from R ;$f(z_k) - z_k = 0$ then $y = z_k x$ $(x>0)$ and $y = z_k x$ $(x<0)$

Point (0,0) is a singular point, and excluded from the domain.

Differential equation in form $y`= f(\frac{ax+by+c}{a_1x+b_1y+c_1})$

Can be solved by the introduction of replacement: $x = u + \alpha$ and $y = v + \beta$, where :

$dx=du$; $dy= dv$ and $v= v(u)$. We are looking for unknown constants α and β.

$\frac{dv}{du} = f(\frac{au+bv+a\alpha+b\beta+c}{a_1u+b_1v+a_1\alpha+b_1\beta+c_1})$, from here must be: $a\alpha + b\beta + c = 0$ and $a_1\alpha + b_1\beta + c_1 = 0$

If $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$ then $a= a_1k$ and $b= b_1k$. If $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$ then $\frac{dv}{du} = g(\frac{v}{u})$ is homogeneous differential equation.

Linear differential equations

Form is: **$y`+ p(x)\, y = q(x)$** and can be solved by formula:

$$y(x) = e^{-\int p(x)dx}\left(c + \int q(x)e^{\int p(x)dx}dx\right)$$

Bernoulli differential equation

Form is: $\mathbf{y' + p(x)\, y = q(x)\, y^n}$

Can be solved by the replacement : $z = y^{1-n}$, then is $z' = (1-n)\, y^{-n}\, y'$.

After replacement we get linear differential equation.

Lagrange differential equations

Form is: $\mathbf{y = xA(y') + B(y')}$

Replacement: $y' = p$, $\frac{dy}{dx} = p$, then is: $dy = pdx$

$y = xA(p) + B(p)$ After differentiation from this equation we get linear differential equation:

$$\frac{dx}{dp} - \frac{A'(p)}{p - A(p)} x = \frac{B'(p)}{p - A(p)}$$

Clero differential equations

Form is: $\mathbf{y = xy' + A(y')}$

Replacement: $y' = p$, $\frac{dy}{dx} = p \longrightarrow dy = pdx$

After differentiation we have: $x + A'(p) = 0$ or $dp = 0$

Riccati differential equations

Form is: $\mathbf{y' = P(x)\, y^2 + Q(x) y + R(x)}$

1) If P,Q,R are constants , then this is differential equation that separates the variables

2) If $y' = Ay^2 + \frac{B}{x} y + \frac{C}{x^2}$ replacement: $z = yx$ where is $z = z(x)$

3) If you know one particular solution $y_1(x)$, then take the replacement: $y(x) = y_1(x) + \frac{1}{z(x)}$ and after that we get linear differential equation.

METHOD with PARAMETERS

If we have function in form F(x,y,y`)=0

1) If y = f(x, y`) then replacement is : y` = p , $\frac{dy}{dx} = p$ ⟶ dy = pdx

dy = $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dp$, in here replace dy = pdx ....

2) If x = g(y,y`) replacement is y` = p , dy= pdx

dx = $\frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial p}dp$ replace dx = $\frac{dy}{p}$ and solve ....

Differential equations with total differential

Form is: P(x,y) + Q(x,y) = 0

Theorem: P(x,y) + Q(x,y) = 0 is differential equation with total differential if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Can be solved by formula: C= $\int P(x,y)dx + \int [Q - \frac{\partial}{\partial y}\int P(x,y)dx]dy$

Integrating factor

If P(x,y) + Q(x,y) = 0 **is not** differential equation with total differential, we are looking for function $\mu = \mu(x,y)$, so the equation μ (x,y) P(x,y)+ μ (x,y) Q(x,y)=0 become differential equation with a total differential. Function $\mu = \mu(x,y)$ is **integrating factor.**

1) If μ (x,y)= $\mu(x)$ then:

$$\int \frac{d\mu}{\mu} = \int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx$$

2) If μ (x,y)= μ (y)

$$\int \frac{d\mu}{\mu} = \int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy$$

3)If μ (x,y)= μ (w(x,y))

$$\int \frac{d\mu}{\mu} = \int \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P\frac{\partial w}{\partial y} - Q\frac{\partial w}{\partial x}}dw$$